College and Career Readiness Standards-in-Action

Foundational Unit

4

WORKSHOP MATERIALS
MATHEMATICS

CONNECTING STANDARDS FOR MATHEMATICAL PRACTICE TO CONTENT

Produced Under U.S. Department of Education
Contract No. ED-VAE-13-C-0066 With StandardsWork, Inc.

2016

TABLE OF CONTENTS

[1] For the purposes of this activity, the lesson from the New York State Education Department Common Core curriculum (EngageNY.org) has been extracted from a complete mathematics module and modified slightly.

Directions for Participants: Part One

1. Carefully read the CCR Level B content standard on the worksheet.

2. Imagine a lesson that targets the standard. Then follow these steps:

    - Use the code X to mark Standards for Mathematical Practice that are central to the lesson you envision.
    - Use the code O to mark Standards for Mathematical Practice that support the lesson you envision.
    - Leave blank any Standards for Mathematical Practice that are not relevant to the lesson you envision.

3. Working first independently and then with a partner, evaluate the relevance of each Standard for Mathematical Practice to the requirements of the content standard listed.

4. Discuss individual decisions and rationales at your table, including how the different types of lessons imagined would affect the relevance of a particular Standard for Mathematical Practice.

Worksheet: Matching Standards for Mathematical Practice to Content Standards

Use the following key to label the Standards for Mathematical Practice below.

- Mark with an **X** each Standard for Mathematical Practice that is likely to be **central** to a lesson that specifically targets the content standard.
- Mark with an **O** each Standard for Mathematical Practice that is more likely to be used in a **supporting role** in a lesson that specifically targets the content standard.
- Leave blank those Standards for Mathematical Practice that are unlikely to be observable in a lesson that targets the content standard.

Provide a rationale for each selected Standard for Mathematical Practice.

<table>
<tr><td colspan="8">CCR Level B Content Standard: Solve two-step word problems using the four operations. Represent these problems using equations with a letter standing for the unknown quantity. Assess the reasonableness of answers using mental computation and estimation strategies including rounding.</td></tr>
<tr><th>MP.1</th><th>MP.2</th><th>MP.3</th><th>MP.4</th><th>MP.5</th><th>MP.6</th><th>MP.7</th><th>MP.8</th></tr>
<tr><td>--</td><td>--</td><td>--</td><td>--</td><td>--</td><td>--</td><td>--</td><td>--</td></tr>
<tr><td colspan="8">Rationales:</td></tr>
</table>

Resource: CCR Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem to gain insight into its solution. They monitor and evaluate their progress, and change course if necessary. Students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Less experienced students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, “Does this make sense?” They can understand the approaches of others to solving complex problems, and identify correspondences between different approaches.

2. Reason abstractly and quantitatively.

Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to *decontextualize*, to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents; and the ability to *contextualize*, to pause as needed during the manipulation process to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

3. Construct viable arguments and critique the reasoning of others.

Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to others’ arguments. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Less experienced students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later. Later, students learn to determine domains to which an argument applies. Students at all levels can listen to or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

4. Model with mathematics.

Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. This might be as simple as writing an addition equation to describe a situation. A student might apply proportional reasoning to plan a school event or analyze a problem in the community. A student might use geometry to solve a design problem, or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts, and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

5. Use appropriate tools strategically.

Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

6. Attend to precision.

Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, and express numerical answers with a degree of precision appropriate in the context of the problem. Less experienced students give carefully formulated explanations to each other. By the time they reach high school, they have learned to examine claims and make explicit use of definitions.

7. Look for and make use of structure.

Mathematically proficient students look closely to discern a pattern or structure. Students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7×8 equals the well-remembered $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, students can see the 14 as 2×7 and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 – 3(x – y)^2$ as 5 minus a positive number times a square, and use that to realize that its value cannot be more than 5 for any real numbers x and y.

8. Look for and express regularity in repeated reasoning.

Mathematically proficient students notice if calculations are repeated, and look for both general methods and shortcuts. Early on, students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, students might abstract the equation $(y – 2)/(x – 1) = 3$. Noticing the regularity in the way terms cancel when expanding $(x–1)(x+1)$, $(x–1)(x^2+x+1)$, and $(x–1)(x^3+x^2+x+1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

Directions for Participants: Part Two

1. Scan the lesson and make notes about how the CCR Standards for Mathematical Practice might be observed in the activities of the lesson.

2. Use the codes X and O to mark the central and supporting Standards for Mathematical Practice. Leave blank any Standards for Mathematical Practice that are not relevant to the requirements of the lesson.

3. Discuss individual decisions and rationales at your table.

Worksheet: Enriching a Mathematics Lesson

Following is the full text of the targeted Level C content standards (as listed in the lesson):

- Explain why a fraction a/b is equivalent to a fraction (n x a)/(n x b) by using visual fraction models, with attention to how the number and size of the parts differ even though the two fractions themselves are the same size. Use this principle to recognize and generate equivalent fractions. **[4.NF.1]**

- Add and subtract fractions with unlike denominators (including mixed numbers) by replacing given fractions with equivalent fractions in such a way as to produce an equivalent sum or difference of fractions with like denominators. *For example, 2/3 + 5/4 = 8/12 + 15/12 = 23/12. (In general, a/b + c/d = (ad + bc)/bd.)* **[5.NF.1]**

Use the following key to label the Standards for Mathematical Practice below.

- Mark with an **X** those Standards for Mathematical Practice that are **central** to the lesson's goals.
- Mark with an **O** those Standards for Mathematical Practice that could be used in a **supporting role.**
- Leave blank those Standards for Mathematical Practice that are unlikely to be observed in either role.

Provide rationales for each selected Standard for Mathematical Practice.

Lesson 1: Equivalent Fractions							
MP.1	**MP.2**	**MP.3**	**MP.4**	**MP.5**	**MP.6**	**MP.7**	**MP.8**
--	--	--	--	--	--	--	--
Rationales:							

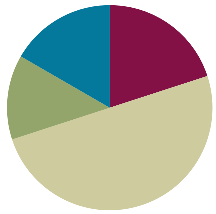
Mathematics – Module 3

Lesson 2 – Equivalent Fractions

Objective: Make equivalent fractions with sums of fractions with like denominators.

Suggested Lesson Structure

Fluency Practice	(12 minutes)
Application Problem	(8 minutes)
Concept Lesson	(30 minutes)
Student Debrief	(10 minutes)
Total Time	**(60 minutes)**



Fluency Practice (12 minutes)

- Equivalent Fractions **5.NF.1** (3 minutes)
- Sprint **4.NF.1** (9 minutes)

Equivalent Fractions (3 minutes)

T: (Write.) $\frac{1}{2} =$

T: Say the fraction.

S: One half.

T: (Write.) $\frac{1}{2} = \frac{}{4}$

T: One half is how many fourths?

S: Two fourths.

Continue with possible sequence:

$$\frac{1}{2} = \frac{}{6}, \frac{1}{3} = \frac{}{6}, \frac{2}{3} = \frac{}{6}, \frac{2}{3} = \frac{}{12}, \frac{3}{4} = \frac{}{16}, \frac{3}{5} = \frac{}{25}$$

T: (Write.) $\frac{1}{2} =$

T: Say the fraction.

S: One half.

T: (Write.) $\frac{1}{2} = \frac{2}{}$

T: One half or one part of two is the same as two parts of what unit?

S: Fourths.

NOTES ON MULTIPLE MEANS OF ENGAGEMENT:

Equivalent Fractions is intentionally placed before the Sprint because it reviews the Sprint skill. Meet the needs of your students by adjusting the amount of time you spend on it. If you find that students struggle to complete Sprint A, you may want to do another minute or two of Equivalent Fractions before moving them on to Sprint B.

NOTES ON MULTIPLE MEANS OF REPRESENTATION:

Adjusting number words and correctly pronouncing them as fractions (fifths, sixths, etc.) may be challenging. If you have many ELLs, before starting you might quickly count together to practice enunciating word endings: hal**ves**, thir**ds**, four**ths**, fif**ths**, six**ths**, etc.

Continue with possible sequence:

$$\frac{1}{2}=\frac{2}{},\ \frac{1}{5}=\frac{2}{},\ \frac{2}{5}=\frac{8}{},\ \frac{2}{5}=\frac{8}{},\ \frac{3}{4}=\frac{9}{},\ \frac{4}{5}=\frac{16}{}$$

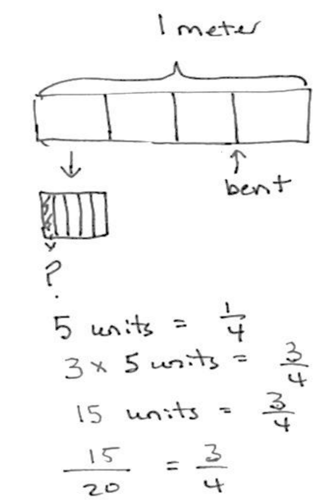
Sprint (9 minutes)

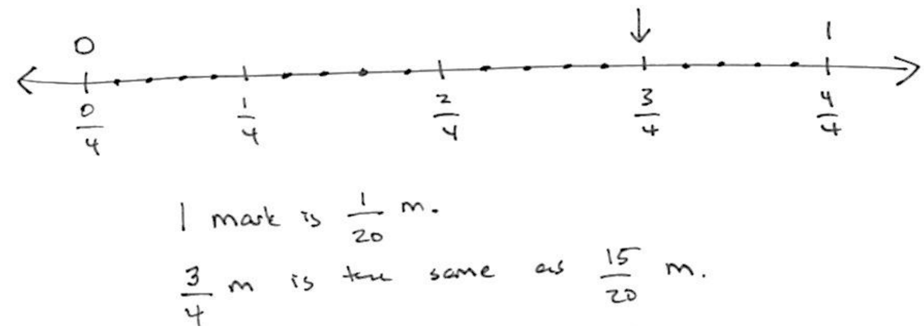
Materials: (S) Find the Missing Numerator or Denominator Sprint

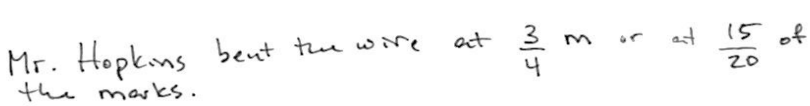
Application Problem (8 minutes)

Mr. Hopkins has a 1-meter wire he is using to make clocks. Each fourth meter is marked off with 5 smaller equal lengths. If Mr. Hopkins bends the wire at ¾ meter, what fraction of the marks is that?

T: (After the students have solved the problem, possibly using the RDW process independently or in partners.) Let's look at two of your solutions and compare them.







T: When you look at these two solutions side by side what do you see? (You might use the following set of questions to help students compare the solutions as a whole class, or to encourage inter-partner communication as you circulate while they compare.)

- What did each of these students draw?
- What conclusions can you make from their drawings?
- How did they record their solutions numerically?
- How does the tape diagram relate to the number line?
- What does the tape diagram/number line clarify?
- What does the equation clarify?
- How could the statement with the number line be rephrased to answer the question?

NOTES ON MULTIPLE MEANS OF REPRESENTATION:

Students have been using the RDW process since Grade 1: Read, Draw, Write.

1. Read the problem.
2. Draw to represent the problem.
3. Write an equation(s) that either helps you to solve the problem or shows how you solved it.
4. Write a statement of the answer to the question.

Embedded within D are important reflective questions:

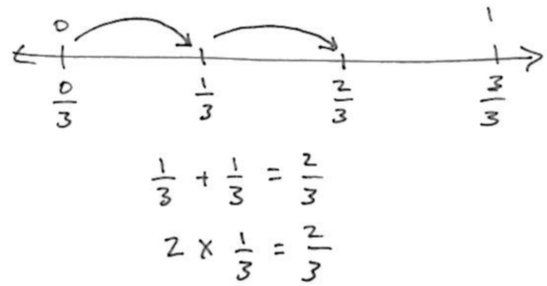
- What do I see?
- Can I draw something?
- What conclusions can I make from my drawing?

Concept Development (30 minutes)

Materials: (S) Blank paper

Problem 1

– – 1 third + 1 third = 2 thirds



T: On a number line, mark the end points as zero and 1. Between zero and 1 estimate to make three parts of equal length and label them with their fractional value.

T: (After students work.) On your number line, show 1 third plus 1 third with arrows designating lengths. (Demonstrate and then pause as students work).

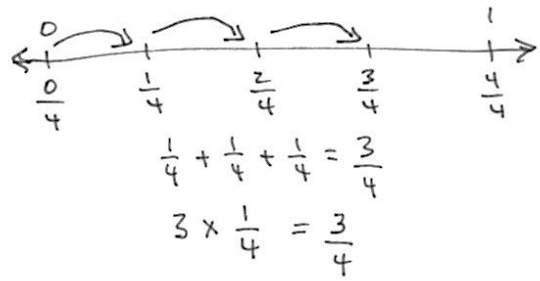
T: The answer is?

S: 2 thirds.

T: Talk to your partner. Express this as a multiplication equation and as an addition sentence.

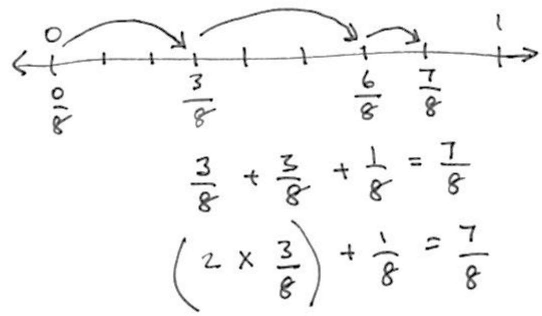
S: – – – –

T: Following the same pattern of adding unit fractions by joining lengths, show 3 fourths on a number line.



Problem 2

– – – 3 eighths + 3 eighths + 1 eighth



T: On a number line, again mark the end points as zero and one. Between zero and one, estimate to make 8 parts of equal length. This time only label what is necessary to show 3 eighths.

T: (After students work.) Represent 3 eighths + 3 eighths + 1 eighth on your number line.

T: The answer is?

S: 7 eighths.

T: Talk to your partner. Express this as a multiplication equation and as an addition equation.

S: – – – – – –

Problem 3

– – – – 6 halves = 3 x 2 halves = 3 ones = 3

T: On a number line, mark the end points as 0 halves and 6 halves below the number line. Estimate to make 6 parts of equal length. This time only label 2 halves.

T: (After students work.) Record the whole number equivalents above the line.

T: Represent 3 x 2 halves on your number line.

T: (After students have worked) The answer is?

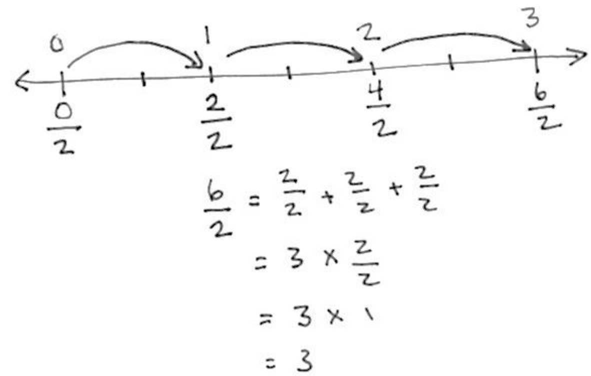
S: 6 halves or 3.

T: 3. What is the unit?

S: 3 ones.

T: Talk to your partner. Express this as an addition equation and as an multiplication equation.

S: – – – – –



Problem 4

– – – – 8 fifths = 5 fifths + 3 fifths = 1 and 3 fifths

T: Use a number line. Mark the end points as 0 fifths and 10 fifths below it. Estimate and give a value to the halfway point.

T: What will be the value of the halfway point?

S: 5 fifths.

T: Make 10 parts of equal length from 0 fifths to 10 fifths.

T: (After students work.) Record the whole number equivalents above the line.

T: (After students work.) Label 8 fifths on your number line.

T: Show 8 fifths as the sum of 5 fifths and 3 fifths on your number line.

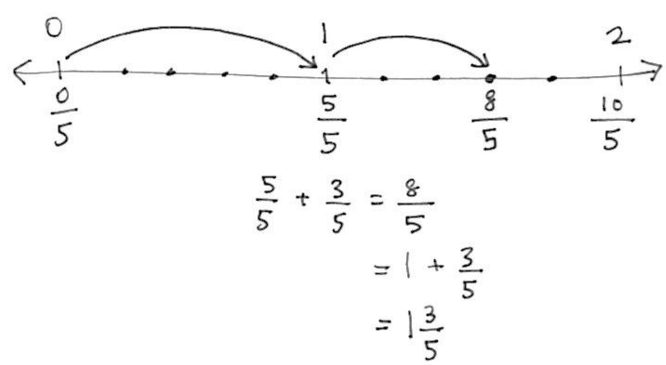
S: (After students work.)

T: Talk to your partner. Express this as an addition equation in two ways: as the sum of fifths and as the sum of a whole number and fifths.

T: (After students work.) Another way of expressing 1 plus 3 fifths is?

S: 1 and 3 fifths.

S: – – – – –

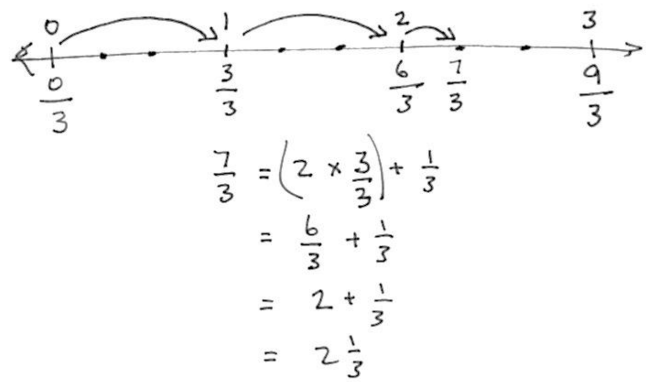


T: 8 fifths is between what 2 whole numbers?

S: 1 and 2.

Problem 5

– – – – – – – 7 thirds = 6 thirds + 1 third = 2 and 1 third.



T: Use a number line. Mark the end points as 0 thirds and 9 thirds below the number line. Divide the whole length into three equal smaller lengths and mark their values using thirds. Work with a partner.

T: (After students work). What are the values of those points?

S: 3 thirds and 6 thirds.

T: Mark the whole number equivalents above the line.

T: (After students work.) Divide each of those whole number lengths into three smaller lengths. Mark the number 7 thirds.

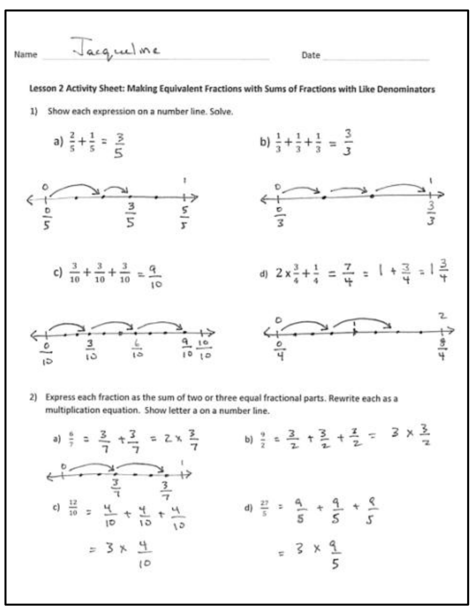
T: (After students work.) Show 7 thirds as two units of 3 thirds and one more third on your number line and in an equation. Work together if you get stuck.

T: (After working and dialogue) 7 thirds is between what two whole numbers?

S: 2 and 3.

Problem Set (10 minutes)

Students should do their personal best to complete the Problem Set within the allotted 10 minutes. For some classes, it may be appropriate to modify the assignment by specifying which problems they work on first. Some problems do not specify a method for solving. Students solve these problems using the RDW approach used for Application Problems.


Name Jacqueline Date

Lesson 2 Activity Sheet: Making Equivalent Fractions with Sums of Fractions with Like Denominators

1) Show each expression on a number line. Solve.

a) $\frac{2}{5}+\frac{1}{5}=\frac{3}{5}$

b) $\frac{1}{3}+\frac{1}{3}+\frac{1}{3}=\frac{3}{3}$

c) $\frac{3}{10}+\frac{3}{10}+\frac{3}{10}=\frac{9}{10}$

d) $2\times\frac{3}{4}+\frac{1}{4}=\frac{7}{4}=1+\frac{3}{4}=1\frac{3}{4}$

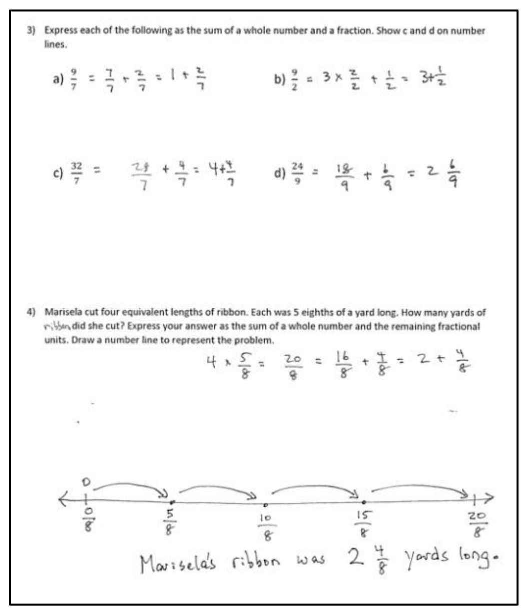
2) Express each fraction as the sum of two or three equal fractional parts. Rewrite each as a multiplication equation. Show letter a on a number line.

a) $\frac{6}{7}=\frac{3}{7}+\frac{3}{7}=2\times\frac{3}{7}$

b) $\frac{9}{2}=\frac{3}{2}+\frac{3}{2}+\frac{3}{2}=3\times\frac{3}{2}$

c) $\frac{12}{10}=\frac{4}{10}+\frac{4}{10}+\frac{4}{10}=3\times\frac{4}{10}$

d) $\frac{27}{5}=\frac{9}{5}+\frac{9}{5}+\frac{9}{5}=3\times\frac{9}{5}$


3) Express each of the following as the sum of a whole number and a fraction. Show c and d on number lines.

a) $\frac{9}{7}=\frac{7}{7}+\frac{2}{7}=1+\frac{2}{7}$

b) $\frac{9}{2}=3\times\frac{2}{2}+\frac{1}{2}=3+\frac{1}{2}$

c) $\frac{32}{7}=\frac{28}{7}+\frac{4}{7}=4+\frac{4}{7}$

d) $\frac{24}{9}=\frac{18}{9}+\frac{6}{9}=2\frac{6}{9}$

4) Marisela cut four equivalent lengths of ribbon. Each was 5 eighths of a yard long. How many yards of ribbon did she cut? Express your answer as the sum of a whole number and the remaining fractional units. Draw a number line to represent the problem.

$4\times\frac{5}{8}=\frac{20}{8}=\frac{16}{8}+\frac{4}{8}=2+\frac{4}{8}$

Marisela's ribbon was $2\frac{4}{8}$ yards long.

Student Debrief (10 minutes)

Lesson Objective: Making equivalent fractions with sums of fractions with like denominators.

The Student Debrief is intended to invite reflection and active processing of the total lesson experience.

Invite students to review their solutions for the Problem Set. They should check work by comparing answers with a partner before going over answers as a class. Look for misconceptions or misunderstandings that can be addressed in the Debrief. Guide students in a conversation to debrief the Problem Set and process the lesson.

T: Come to the Debrief and bring your Problem Set. Compare your work to your neighbor's. On which problems do you have different answers? Discuss your differences. Both may be correct.

T: (After about 3 minutes.)

T: What is a way to express 3/7 as a sum?

S: 1 sevenths + 1 seventh + 1 seventh.

T: Another way?

S: 2 sevenths + 1 seventh.

T: These are equivalent forms of 3 sevenths.

T: On your Problem Set find and talk to your partner about different equivalent forms of your numbers.

S: 6 sevenths could be expressed as 3 sevenths + 3 sevenths or as 3 times 2 sevenths. → 9 sevenths can be expressed as 1 + 2 sevenths. → 7 fourths can be expressed as 2 times 3 fourths + 1 fourth. → 1 and 3 fourths can be expressed as 7 fourths. → 32 sevenths can be expressed as 28 sevenths + 4 sevenths or 4 and 4 sevenths.

T: I'm hearing you express these numbers in many equivalent forms. Why do you think I chose to use the tool of the number line in this lesson? Talk this over with your partner. If you were the teacher of this lesson, why might you use the number line?

T: (After students discuss.) When we were studying decimal place value, we saw that 9 tenths + 3 tenths is equal to 12 tenths or 1 + 2 tenths or 1 and 2 tenths.

T: Once more, please review the solution and number line you made for question 4 about Marisela's ribbon. Discuss the equivalence of 20 eighths and 2 and 4 eighths as it relates to the number line.

T: (After students talk.) Discuss the relationship of the equivalence of these sums.

9 tenths + 3 tenths = 12 tenths = 1 + 2 tenths = 1 —.

9 elevenths + 3 elevenths = 12 elevenths = 1 + 1 eleventh = 1 —.

T: (After students talk.) Yes, our place value system is another example of equivalence.

Exit Ticket (3 minutes)

After the Student Debrief, instruct students to complete the Exit Ticket. A review of their work will help you assess the students' understanding of the concepts that were presented in the lesson today and plan more effectively for future lessons. You may read the questions aloud to the students.

Answer Key for Part One: Matching Standards for Mathematical Practice to Content Standards

NOTE: These sample rationales are just examples and are not the only correct answers.

Level B Content Standard: Solve two-step word problems using the four operations. Represent these problems using equations with a letter standing for the unknown quantity. Assess the reasonableness of answers using mental computation and estimation strategies including rounding.							
MP.1	**MP.2**	**MP.3**	**MP.4**	**MP.5**	**MP.6**	**MP.7**	**MP.8**
X	**O**	**--**	**O**	**--**	**X**	**--**	**--**
Sample Rationale: *A lesson targeting this standard will likely use problem-solving as its primary target, as several of the skills in the MP.1 description are specifically cited. Since requirements regarding attention to the accuracy of the results are explicitly addressed in the standard, MP.6 is also likely to be selected as central to a lesson targeting this standard. In addition, since there is an explicit requirement to represent word problems using equations, it is likely that participants will select MP.4 in a supporting role. Another possibility of a supporting Standard for Mathematical Practice is MP.2, since the requirement to use variables to represent the problem is a form of decontextualizing, and assessing the reasonableness of the answer is recontextualizing.*							

Answer Key for Part Two: Enriching a Mathematics Lesson

Full text of the targeted Level C content standards (as listed in the lesson):

- Explain why a fraction a/b is equivalent to a fraction (n x a)/(n x b) by using visual fraction models, with attention to how the number and size of the parts differ even though the two fractions themselves are the same size. Use this principle to recognize and generate equivalent fractions. **[4.NF.1]**

- Add and subtract fractions with unlike denominators (including mixed numbers) by replacing given fractions with equivalent fractions in such a way as to produce an equivalent sum or difference of fractions with like denominators. *For example, 2/3 + 5/4 = 8/12 + 15/12 = 23/12. (In general, a/b + c/d = (ad + bc)/bd.)* **[5.NF.1]**

NOTE: These sample rationales are just examples and are not the only correct answers.

Lesson: Equivalent Fractions							
MP.1	**MP.2**	**MP.3**	**MP.4**	**MP.5**	**MP.6**	**MP.7**	**MP.8**
--	**X**	--	--	--	**O**	**X**	--

Sample Rationales:
MP.2: *An important aspect of this lesson is interpreting the denominator value of the fractions as a unit. This is a way of contextualizing the structure of the fraction itself.*
MP.6: *In this lesson, attention is paid to distances on number lines and finding accurate equivalences. This indicates a supporting role for "attending to precision."*
MP.7: *Participants might consider that as students begin to understand how to find equivalent fractions, they need to pay attention to the structure of the fractions.*